Enhancing chaos in chaotic maps and flows

Neelima Gupte and R. E. Amritkar
Department of Physics, University of Poona, Pune 411007, India
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We propose a mechanism by which the degree of chaoticity as measured by the average Lyapunov exponent in chaotic flows can be increased. Our mechanism consists of introducing small changes in the system parameters in regions of phase space where the local Lyapunov exponent falls substantially below its average value. We have applied our mechanism to several typical chaotic flows and maps, dissipative as well as area preserving. An interesting consequence of this increase in chaoticity is an enhancement of the rate of mixing of the system. We find that our method is quite efficient as it gives a substantial enhancement of chaos as measured by the average Lyapunov exponent and also the rate of mixing for small changes in system parameters. [S1063-651X(96)01510-3]

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I. INTRODUCTION

Most studies which attempt to control chaotic dynamical systems direct their efforts towards controlling the system to regular periodic orbits or to specific chaotic orbits [1,2]. However, there have been few attempts at control directed towards enhancing the chaoticity of chaotic flows. This is an important problem for its own intrinsic interest and may have practical applications as well. An important example of a situation where enhancing chaos is useful, is the process of mixing [3–5]. Mixing is a consequence of the stretching and folding of chaotic flows. A system which has exponential stretching, as in a chaotic flow, can mix efficiently. Many mixing processes like fluid flows, combustion processes, chemical reactions, heat transfer processes, etc., can be modeled by chaotic flows [3]. An enhancement of the chaoticity of such systems can lead to an enhancement of the rate of mixing, an outcome which has desirable consequences in many of these contexts. In addition to enhancing the rate of mixing, the enhancement of chaos can be desirable and useful in other situations also . In the case of biological systems, there are several instances of situations where maintaining or enhancing chaos is desirable [6]. It has been suggested that the pathological destruction of chaotic behavior may be responsible for heart failure [7], and some types of brain seizures [8]. Techniques which are capable of enhancing and maintaining chaos could be useful in such contexts [9]. We propose a mechanism to enhance chaos in this paper.

An important parameter, which characterizes the degree of chaos in a chaotic flow is the Lyapunov exponent, which gives the average rate of stretching. However, the rate of stretching is not uniform over a chaotic attractor in the case of dissipative flows or over the phase space of a conservative flow. Thus the local Lyapunov exponent (LLE), a measure of the local rate of stretching, is different in different regions of the phase space [10]. We exploit the nonuniform nature of the spatial distribution of the LLEs to construct a mechanism that can enhance chaos and, hence, the rate of chaotic mixing. Briefly, we enhance the average rate of stretching by introducing a small parameter perturbation which enhances the LLE whenever the system trajectory visits a region where the LLEs take values much smaller than their average value. We find that this procedure works quite efficiently as small perturbations in parameter made for small times compared to the total time of evolution can lead to substantial enhancement of the Lyapunov exponent and to the rate of mixing. Our examples are both dissipative and conservative in nature and discuss mixing in phase space. However, our conservative examples are relevant to mixing in real space in the same sense in which the Lagrangian description of a two-dimensional fluid defined in physical space by the fluid velocity field obtainable from a stream function can be mapped on to a Hamiltonian flow in two-dimensional phase space [11].

II. THE MECHANISM

Let us consider an autonomous nonlinear dynamical system $\mathbf{x}$ of dimension n, evolving via the equations

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mu), \tag{1}$$

where the set of parameters μ takes values such that the trajectory shows chaotic behavior. Let $\mathbf{w}(\mathbf{x},t)$ be the tangent vector to the trajectory at the point $\mathbf{x}$ and time t. The evolution of $\mathbf{w}$ is given by

$$\dot{\mathbf{w}}=(\mathbf{w}\cdot\nabla)\mathbf{F}. \tag{2}$$

The Lyapunov exponent of the system is defined by

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{||\mathbf{w}(\mathbf{x},t)||}{||\mathbf{w}(\mathbf{x}(0),0)||}, \tag{3}$$

where $\mathbf{x}(0)$ is the value of $\mathbf{x}$ at $t=0$ and $||\mathbf{w}||$ is the norm of $\mathbf{w}$. We now define the local Lyapunov exponent $\lambda(\mathbf{x})$ as

$$\lambda(\mathbf{x})=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\ln\frac{||\mathbf{w}[\mathbf{x}(t+\Delta t),t+\Delta t]||}{||\mathbf{w}[\mathbf{x}(t),t]||}. \tag{4}$$

The quantity $\lambda(\mathbf{x})$ represents the local rate of stretching at the point $\mathbf{x}$. This is, in general, not uniform over the attractor. We also note that the Lyapunov exponent λ [Eq. (3)] is the average value of the LLEs for a long orbit or can be obtained by averaging the LLEs over the invariant density of the attractors of dissipative systems.

We set up a control procedure to enhance chaos and insofar as this improves mixing, increase the mixing rate utilizing the distribution of the LLEs. The control procedure operates in regions where the LLEs fall substantially below the average value λ. If, at any time, the LLE of the system falls below its average value to the point where

$$\lambda(\mathbf{x}) < (\lambda - \gamma\sigma_\lambda), \tag{5}$$

where σ_λ is the standard deviation of the distribution of LLE and γ is some chosen factor, the control is activated so that the parameter μ is changed to $\mu + s d\mu$. Here $d\mu$ is a small increment and s takes values $+1$ or -1 depending on which choice enhances the LLE. The system is allowed to evolve with the new value of the parameter as long as the condition (5) is satisfied. Thereafter the parameter is reset to its original value.

To decide the sign s, we write an equation for $\mathbf{w}$ in matrix notation in the form

$$\dot{W}^T = W^T M^T, \quad \dot{W} = M W, \tag{6}$$

where W^T is a row vector and the matrix M^T is given by $M^T = \nabla \mathbf{F}$. The equation for the norm of W can be written as

$$||\dot{W}||^2 = W^T (M^T + M) W. \tag{7}$$

Thus the rate of change in the norm of W due to change in the parameter is given by

$$\Delta ||\dot{W}||^2 = ||\dot{W}(\mu + d\mu)||^2 - ||\dot{W}(\mu)||^2$$
$$\simeq W^T (M_\mu^T + M_\mu) W d\mu, \tag{8}$$

where the last step is obtained by expanding to lowest order in $d\mu$ and $M_\mu = \partial M / \partial \mu$. Clearly, for the local rate of stretching to increase $\Delta ||W||^2$ must be positive . Thus the sign s is determined to ensure that $\Delta ||W||^2$ is positive.

It must be noted that Eq. (8) is written in the lowest order in $d\mu$. Actually, the effect of the perturbation is nonlinear since when the parameter changes the entire trajectory of the system changes. Hence, the effect on the LLE can be quite different from that given by Eq. (8) due to the effect of the higher nonlinear terms. In many cases the enhancement in the Lyapunov exponent turns out to be substantially higher than that expected in the linear approximation.

The procedure used above to enhance chaos and the mixing rate can be easily modified to apply to the case of discrete maps. For maps, the evolution equation [Eq. (1)] can be written as

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, \mu), \tag{9}$$

where $\mathbf{x}_t$ are the dynamical variables at time t. The evolution of the tangent vector $\mathbf{w}$ is given by

$$\mathbf{w}_{t+1} = (\mathbf{w}_t \cdot \nabla)\mathbf{f}. \tag{10}$$

The control procedure is the same as above. The parameter μ is changed to $\mu + s d\mu$ when condition (5) is satisfied. To decide the sign s we write Eq. (10) in matrix form as

$$W_{t+1} = M W_t, \tag{11}$$

where $M^T = \nabla \mathbf{f}$. The equation for the norm of W is

$$||W_{t+1}||^2 = W_t^T M^T M W_t. \tag{12}$$

Thus the rate of change in the norm of W due to change in the parameter is given by

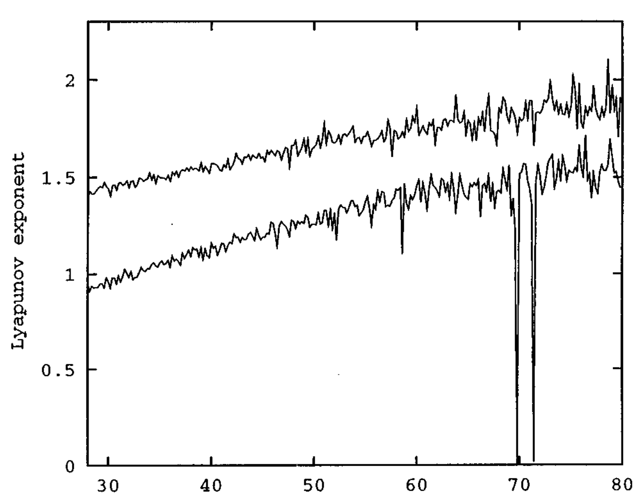

FIG. 1. The plot of the Lyapunov exponent of the Lorenz attractor for the parameters $\sigma = 10.0$, $b = 2.6666$, and r from $r = 28.0$ to $r = 80.0$. The lower curve and upper curves correspond to the Lyapunov exponent for the uncontrolled and controlled systems, respectively with $\gamma = 0.5$ and $dr = 1.0$.

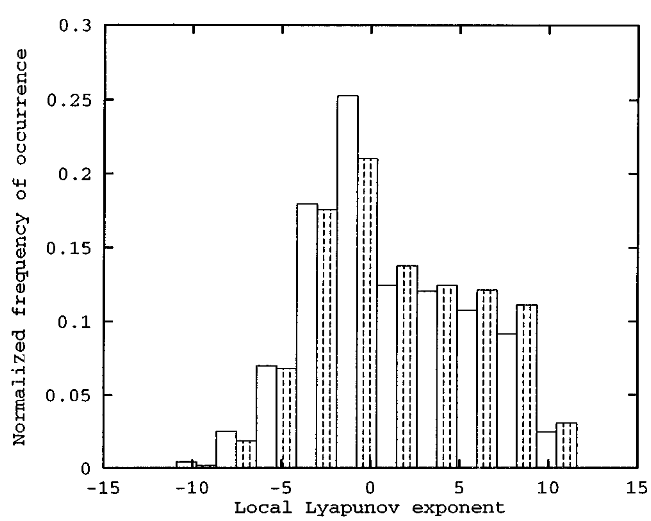

FIG. 2. A histogram of the distribution of the LLEs of the uncontrolled and controlled Lorenz systems for the parameter values $\sigma = 10.0$, $r = 30.0$, $b = 2.6666$, $\gamma = 0.5$, and $dr = 1.0$. The data has been binned into ten boxes. The bar without lines shows the normalized frequency of occurrence of the corresponding LLE of the uncontrolled system and the bar with vertical lines shows the same quantity for the controlled system.

TABLE I. We list the uncontrolled (free) and controlled (cont.) values of the Lyapunov exponent and of the fractal dimensions (calculated by the box-counting algorithm) for several dissipative maps and flows. The values of the parameters of the systems analyzed are listed in the text. The column Fract. refers to the fraction of time for which the system is controlled and $d\mu$ is the parameter change.

System	γ	$d\mu$	Fract.	Lyapunov exp.		Dimension	
				Free	Cont.	Free	Cont.
Lorenz	0.5	$dr=1.0$	0.344	0.951	1.440	2.052	2.056
	0.25	$dr=1.0$	0.447	0.951	1.362	2.052	2.061
Williamowski -Rössler	1.0	$dk_1=1.5$	0.047	0.559	0.804	2.069	2.068
Hénon	0.5	$da=0.1$	0.263	0.306	0.328	1.206	1.212

$$\Delta\|W_{t+1}\|^2=\|W_{t+1}(\mu+d\mu)\|^2-\|W_{t+1}(\mu)\|^2 \quad (13)$$

$$=W_t^T(M^TM_\mu+M_\mu^TM)W_t d\mu, \quad (14)$$

where the last step is obtained by expanding to lowest order in $d\mu$ and $M_\mu=\partial M/\partial\mu$. For control to enhance chaos and rate of mixing, the sign s for the parameter change $d\mu$ must be such that $\Delta\|W_{t+1}\|^2$ is positive. We shall illustrate our procedure for both flows and maps in the next section.

III. ILLUSTRATIONS

A. Flows

We now illustrate our procedure using some typical flows. We first consider the Lorenz system [12] given by

$$\dot{x}=\sigma(y-x), \quad \dot{y}=rx-y-xz, \quad \dot{z}=xy-bz. \quad (15)$$

We choose r as the control parameter. The perturbation is switched on when the condition (5) is satisfied. The sign of the perturbation dr is obtained using Eq. (8) and the sign is decided by $W_xW_ydr>0$. For a change of parameter $dr=1.0$, $\gamma=0.5$, and with parameters $\sigma=10.0, r=30.0$, and $b=8/3$. The Lyapunov exponent of the system is enhanced from $\lambda=0.950$ for the uncontrolled case to $\lambda=1.440$ (See Table I).

This enhancement in the Lyapunov exponent is not confined to the parameter values above. The plot of the Lyapunov exponent of the system as a function of r for both the uncontrolled and the controlled case is shown in Fig. 1. It is clear from the figure that there is a substantial enhancement of the Lyapunov exponent over the entire range plotted in the figure. It can also be seen that the control mechanism has succeeded in producing a positive Lyapunov exponent in a parameter regime where the uncontrolled system has a periodic window. Thus the technique has been successful in both enhancing and maintaining chaos. This enhancement has been effected by causing a change in the LLEs of the system via parameter change. To show this, we plot the distribution of the LLE for $r=30.0$ for both the uncontrolled and the controlled cases in Fig. 2. It is clear that the distribution of local Lyapunov exponents of the system has changed in a manner in which the average exponent is significantly enhanced.

In order to show that the increase in the Lyapunov exponent leads to an enhancement of the rate of mixing, we operated the control procedure on a large number of initial conditions in a small region of phase space. We cover the attractor with a grid of cubic boxes. We take a large number of initial conditions in a randomly chosen box. Each initial condition evolves with a distinct trajectory and the control is operative for a given trajectory (corresponding to a given initial condition) whenever the LLE of the trajectory satisfies condition (5). The initial conditions are also evolved separately without the control. The initial conditions initially in one box spread over several boxes with time. A comparison of the number of occupied boxes, i.e., the boxes which have at least one initial condition, as a function of time for the uncontrolled and the controlled systems gives us an idea of the relative rates of mixing of the two systems. Figure 3 plots the number of occupied boxes as a function of time for both the uncontrolled and the controlled systems with a grid of 10^3 boxes and 10^5 initial conditions.The parameter values are $\sigma=10.0$, $r=30.0$, $b=2.6666$, $dr=1.0$, and $\gamma=0.25$ [13]. It is clear from the figure that the controlled system mixes at a faster rate than the uncontrolled one. The results are unchanged for any randomly chosen initial box. This demonstrates that the control procedure has successfully enhanced the rate of mixing of the system.

We have also applied our algorithm to the Williamowski-

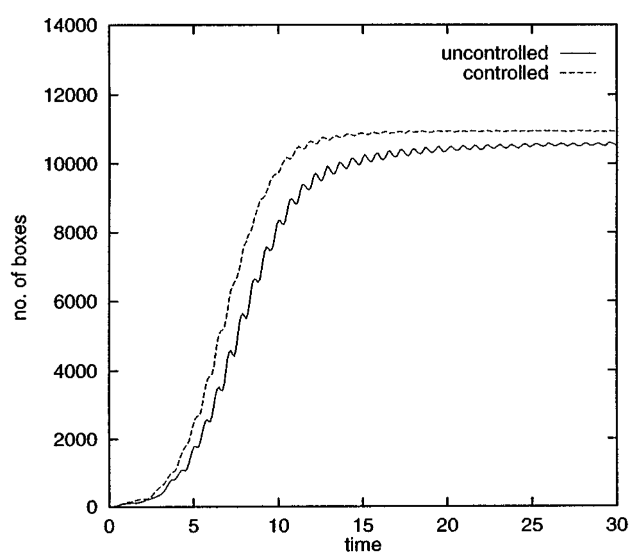

FIG. 3. The plot of the number of occupied boxes as a function of time for the uncontrolled (solid line) and controlled (dashed line) Lorenz systems. The parameter values are $\sigma=10.0$, $b=2.666\,66$, $r=30.0$, $\gamma=0.25$, and $dr=1.0$.

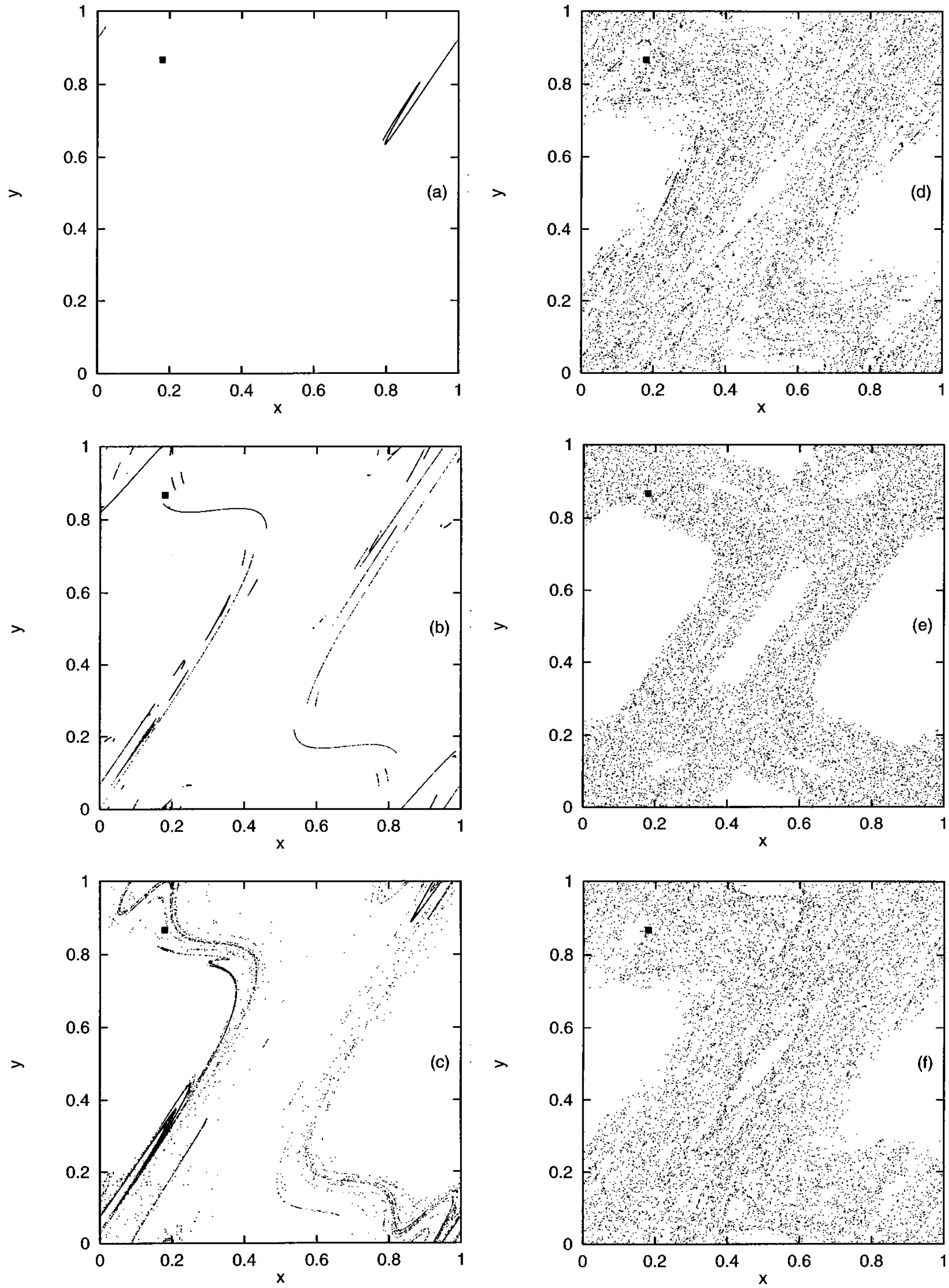

FIG. 4. We show the spread of 10 000 points, initially in the same box of a 64×64 grid on the phase space of the standard map. (a) and (b) show the uncontrolled and controlled standard map after 10 iterates, and (c) and (d) show similar graphs after 30 iterates. The parameter values are $K=1.5$, $dK=0.1$, and $\gamma=0.25$. The solid box seen in each figure is the set of initial conditions that are being iterated. The effect of the control procedure on global mixing is demonstrated in (e) and (f) where uncontrolled and controlled points are, respectively, shown after 200 iterates.

Rössler attractor, which models a system of chemical reactions [14,15]. The Williamowski-Rössler system evolves via the system of equations

$$\dot{x}=k_1x-k_{-1}x^2-k_2xy+k_{-2}y^2-k_4xz+k_{-4},$$

$$\dot{y}=k_2xy-k_{-2}y^2-k_3y+k_{-3},$$

$$\dot{z}=-k_4xz+k_{-4}+k_5z-k_{-5}z^2. \tag{16}$$

The system is allowed to evolve at the parameter values $k_1=30.0$, $k_2=1.0$, $k_3=10.0$, $k_4=1.0$, $k_5=16.5$, $k_{-1}=0.25$, $k_{-2}=1.0\times10^{-4}$, $k_{-3}=1.0\times10^{-3}=k_{-4}$, $k_{-5}=0.5$. Control is effected via a change in parameter k_1 whenever the condition (5) is satisfied. As seen from Table I this results in a large enhancement of the Lyapunov exponent from the uncontrolled value $\lambda=0.559$ to the value $\lambda=0.804$ after the application of the control. We have also verified that the rate of mixing is enhanced due to the control by evolving a large number of initial conditions in a small region of phase space.

B. Maps

Consider the Hénon map [16] given by

$$x_{t+1}=1-ax_t^2+y_t, \quad y_{t+1}=by_{t+1}. \tag{17}$$

If our control procedure is applied to the Hénon map at parameter values $a=1.2$, $b=0.3$ with $da=0.1$ as the parameter change, we again find an enhancement of the Lyapunov exponent from the uncontrolled value $\lambda=0.306$ to the controlled value $\lambda=0.328$ (see Table I). Other values of the parameters also give similar results.

All our examples so far have dealt with dissipative flows. However, our procedure is completely general and can be applied to conservative systems as well, provided the change in parameter is made in a way in which the conservative nature of the flow is preserved. We demonstrate the increase in chaos and rate of mixing in the standard map [11].

$$x_{t+1}=x_t+\frac{K}{2\pi}\sin(2\pi y_t), \quad y_{t+1}=y_t+x_{t+1}. \tag{18}$$

The increase in the rate of spread of initial conditions in phase space of the controlled standard map as compared to the uncontrolled one is demonstrated in Fig. 4. The parameter values are $K=1.5$, $dK=0.2$, $\gamma=0.25$. It can be easily seen that the controlled map retains its area-preserving property. We iterate 10 000 initial conditions initially in a small box. Figures 4(a) and 4(b) show the uncontrolled and controlled systems, respectively, after ten iterations. Figs. 4(c) and 4(d) show the corresponding figures after 30 iterations. It is clear that the controlled system has spread out more than the uncontrolled situations. Thus the rate of mixing is enhanced. The Lyapunov exponent for the orbit with the initial condition $x=0.2$, $y=0.3$ increases from $\lambda=0.286$ to $\lambda=0.384$. The invariant KAM tori in the phase space of the standard map function as barriers to global mixing. To see the effect of control on asymptotic global mixing we plot the uncontrolled and controlled system after 200 iterates in Figs. 4(e) and 4(f), respectively. We find that the attractor is very similar in both the cases except that the controlled attractor spreads more. The area of empty regions in the uncontrolled situation [Fig. 4(e)] is larger than the area of empty regions in the controlled situation [Fig. 4(f)]. We also plot the number of occupied boxes as a function of time for both the controlled and uncontrolled standard maps in Fig. 5. It is clear that the number of occupied boxes grows much faster for the controlled system.

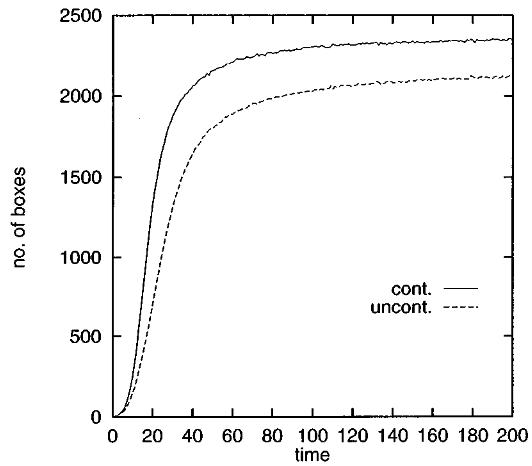

FIG. 5. The plot of the number of occupied boxes as a function of time for the controlled (solid line) and uncontrolled (dashed line) standard maps. The parameter values are $K=1.5$, $dK=0.2$. The curves are obtained after averaging over 25 different initial boxes.

IV. DISCUSSION AND CONCLUSIONS

As demonstrated above, our control procedure works for all the maps and flows tested and the Lyapunov exponent is substantially enhanced in most cases. The Lyapunov exponent referred to in the entire discussion above is the largest Lyapunov exponent of the system. It is also interesting to look at the other Lyapunov exponents of the system. For the Lorenz attractor, for the parameter values $\sigma=10.0$, $r=30.0$, $b=2.6666$, $dr=1.0$, and $\gamma=0.5$, the complete set of Lyapunov exponents of the uncontrolled system is given by (0.950, 0.000, -14.617), and that of the controlled system is given by (1.440, -0.420, -14.686). Thus the largest Lyapunov exponent of the system, which is a measure of the rate of stretching, has increased whereas the other Lyapunov exponents of the system, have become more negative signifying an increase in the rate of contraction. The controlled system no longer has a zero Lyapunov exponent as the trajectory is no longer smooth. These observations are also true of the other dissipative systems studied. The control procedure tends to push the trajectory in the basin of attraction of the uncontrolled attractor, but the increased rate of contraction pushes it back to the original attractor. Both these factors work to our advantage. As mentioned earlier, the increase in the rate of stretching tends to mix the system better. The increase in the rate of contraction has the advantage that it tends to stabilize the attractor. This works best for dissipative systems and is the origin of the insignificant change in the fractal dimension of the controlled and uncontrolled attractors in these cases (see Table I).

The control procedure outlined above leads to the enhancement of chaos and the rate of mixing for most parameter settings. However, we did find a few cases where it did not work well, e.g., in the neighborhood of the parameter values $r=138.0$ and $r=160.0$ for the Lorenz attractor, where there is a wide periodic window nearby. In such cases the control tends to push the trajectory in the neighborhood of a periodic orbit resulting in intermittent behavior and can even lead to a decrease in the Lyapunov exponent. This problem can be easily taken care of by changing the magnitude of the parameter change and/or the factor γ.

Yet another problem can arise in the case of conservative systems. As seen in the case of the standard map, our control mechanism improves the rate of mixing in the ergodic regions of the phase space, and the controlled system spreads more in phase space than the uncontrolled one. However, there still remain barriers to global mixing in the form of invariant KAM curves in the phase space. It may be possible to solve this problem in some cases by an appropriate choice of the factor γ and the magnitude of the parameter change. However, this may not always be possible.

Our control procedure works quite efficiently as it produces a substantial enhancement of the Lyapunov exponent for quite small changes in the parameters. This is due to the fact that our control procedure works by switching between three types of chaotic flows, those characteristic of parameter values μ, $\mu+d\mu$ and $\mu-d\mu$. This switching introduces an extra time dependence in the problem and is the origin of the efficiency of the procedure.

To summarize, we have introduced an efficient mechanism that produces a substantial increase in the chaos and the rate of mixing for small changes in parameters. This mechanism enhances the degree of chaoticity of the system and hence can be useful in contexts other than the mixing context emphasized above. The success of the mechanism has been demonstrated for several chaotic flows and maps. We hope that this mechanism will prove to be useful in enhancing the rate of mixing and also in other practical contexts.

ACKNOWLEDGMENTS

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